Inequality
https://www.linkedin.com/groups/8313943/8313943-6420989678299672579
Let $x_1,x_2,x_3,\ldots,x_n$ be positive real numbers such that
$x_1x_2x_3\ldots x_n=1$, prove that
$\dfrac{1}{1+x_1+x_1x_2}+\dfrac{1}{1+x_2+x_2x_3}+\ldots+\dfrac{1}{1+x_n+x_nx_1}>1.$

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First we will prove inequality of the problem for two particular cases.
If $n=3$ then since $x_1x_2x_3=1$ we obtain

$$\frac{1}{1+x_1+x_1x_2}+\frac{1}{1+x_2+x_2x_3}+\frac{1}{1+x_3+x_3x_1}=$$
$$\frac{1}{1+x_1+x_1x_2}+\frac{x_1}{x_1(1+x_2+x_2x_3)}+\frac{x_1x_2}{x_1x_2(1+x_3+x_3x_1)}=$$
$$\frac{1}{1+x_1+x_1x_2}+\frac{x_1}{x_1+x_1x_2+1}+\frac{x_1x_2}{x_1x_2+1+x_1}=1;$$

If $n=3$ then since $x_1x_2x_3x_4=1$ we obtain

$$\frac{1}{1+x_1+x_1x_2}+\frac{1}{1+x_2+x_2x_3}+\frac{1}{1+x_3+x_3x_4}+\frac{1}{1+x_4+x_4x_1}>$$
$$\frac{1}{1+x_1+x_1x_2+x_1x_2x_3}+\frac{1}{1+x_2+x_2x_3+x_2x_3x_4}+$$
$$\frac{1}{1+x_3+x_3x_4+x_3x_4x_1}+\frac{1}{1+x_4+x_4x_1+x_4x_1x_2}=\frac{1}{1+x_1+x_1x_2+x_1x_2x_3}+$$
$$\frac{x_1}{x_1(1+x_2+x_2x_3+x_2x_3x_4)}+\frac{x_1x_2}{x_1x_2(1+x_3+x_3x_4+x_3x_4x_1)}+$$
$$\frac{x_1x_2x_3}{x_1x_2x_3(1+x_4+x_4x_1+x_4x_1x_2)}=\frac{1+x_1+x_1x_2+x_1x_2x_3}{1+x_1+x_1x_2+x_1x_2x_3}=1$$

General case.

For any n –tipple $(x_1,x_2,\ldots,x_n)$ denote $\sigma(x_1,x_2,\ldots,x_n):=x_2,x_3,\ldots,x_n,x_1$ (Operator of cyclic shift).

Also we recursively define k –times iterated cyclic shift as follows

$\sigma^1:=\sigma,\ \sigma^{k+1}:=\sigma\circ\sigma^k, k\in\mathbb{N}$ while noting that $\sigma^{k+n}(x_1,x_2,\ldots,x_n)=\sigma^k(x_1,x_2,\ldots,x_n)$,

We can extend definition of σ^k for $k=0$ by $\sigma^0(x_1,x_2,\ldots,x_n):=(x_1,x_2,\ldots,x_n)$ $(\sigma^n=\sigma^0)$.

Then for any function $f(x_1,x_2,\ldots,x_n)$ we can use convenient notation of cyclic summation:

$$\sum_{cyc}^{n}f(x_1,x_2,\ldots,x_n):=\sum_{k=1}^{n}\sigma^{k-1}(x_1,x_2,\ldots,x_n)=f(x_1,x_2,\ldots,x_n)+f(x_2,x_3,\ldots,x_n,x_1)+$$
$$f(x_3,\ldots,x_n,x_1,x_2)+\ldots+f(x_n,x_1\ldots,x_{n-1}).$$

Obvious that $\sum_{cyc}^{n}f(x_k,x_{k+1},\ldots,x_n,x_1,x_2,\ldots x_{k-1})=\sum_{cyc}^{n}f(x_1,x_2,\ldots,x_n)$.

In this notation original inequality becomes $\sum_{cyc}^{n}\dfrac{1}{1+x_1+x_1x_2}>1$.

Note that $\dfrac{1}{1+x_1+x_1x_2}>\dfrac{1}{1+x_1+x_1x_2+x_1x_2x_3+\ldots+x_1x_2\ldots x_{n-1}}$.

Let $S(x_1,x_2,\ldots,x_n):=x_1+x_1x_2+x_1x_2x_3+\ldots+x_1x_2\ldots x_{n-1}+x_1x_2\ldots x_{n-1}x_n=$
$1+x_1+x_1x_2+x_1x_2x_3+\ldots+x_1x_2\ldots x_{n-1}$ since $x_1x_2x_3\ldots x_n=1$.

Then $\dfrac{1}{1+x_1+x_1x_2}>\dfrac{1}{S(x_1,x_2,\ldots,x_n)}$ and

$$\frac{1}{1+x_k+x_kx_{k+1}} > \frac{1}{S(x_k,x_{k+1},\dots,x_n,x_1,x_2,..x_{k-1})} = \frac{1}{S(\sigma^{k-1}(x_1,x_2,\dots,x_n))}$$

We have $\frac{S(x_1,x_2,\dots,x_n)}{x_1} = 1+x_2+x_2x_3+\dots+x_2x_3\dots x_{n-1}x_n =$

$x_2+x_2x_3+\dots+x_2x_3\dots x_{n-1}x_n+x_2x_3\dots x_{n-1}x_nx_1 = S(x_2,x_3,\dots,x_n,x_1) = S(\sigma(x_1,x_2,\dots,x_n))$.

Similarly, $\frac{S(x_2,x_3,\dots,x_n,x_1)}{x_2} = S(\sigma(x_2,x_3,\dots,x_n,x_1)) = S(\sigma^2(x_1,x_2,\dots,x_n))$, and so on..

$\frac{S(\sigma^{k-1}(x_1,x_2,\dots,x_n))}{x_k} = \frac{S(x_k,x_{k+1},\dots,x_n,x_1,x_2,..,x_{k-1})}{x_k} = S(\sigma^k(x_1,x_2,\dots,x_n)),..$

Thus, $S(\sigma^k(x_1,x_2,\dots,x_n))\cdot x_1x_2\dots x_k = S(x_1,x_2,\dots,x_n)$ and, therefore,

$$\sum_{cyc}^{n}\frac{1}{1+x_1+x_1x_2} > \sum_{k=1}^{n}\frac{1}{S(\sigma^{k-1}(x_1,x_2,\dots,x_n))} = \frac{1}{S(x_1,x_2,\dots,x_n)} +$$

$$\sum_{k=2}^{n}\frac{x_1x_2\dots x_{k-1}}{S(\sigma^{k-1}(x_1,x_2,\dots,x_n))\cdot x_1x_2\dots x_{k-1}} = \frac{1}{S(x_1,x_2,\dots,x_n)} + \sum_{k=2}^{n}\frac{x_1x_2\dots x_{k-1}}{S(x_1,x_2,\dots,x_n)} =$$

$$\frac{x_1x_2x_3\dots x_n+\sum_{k=2}^{n}x_1x_2\dots x_{k-1}}{S(x_1,x_2,\dots,x_n)} = \frac{S(x_1,x_2,\dots,x_n)}{S(x_1,x_2,\dots,x_n)} = 1$$